

# Neutrino Masses and Mixing Angles in a Supersymmetric $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ Model

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## Abstract

We consider the problem of neutrino masses and mixing angles in a supersymmetric model based on the gauge group  $SU(4) \otimes SU(2)_L \otimes SU(2)_R$  broken at the scale  $M_X \approx 10^{16}$  GeV. We extend a previous operator analysis of the charged lepton and quark masses and mixing angles in this model to include the neutrino sector, assuming a universal Majorana mass  $M$  for the right-handed neutrinos. The Dirac part of the neutrino matrix is then fixed and the physical neutrino masses and magnitudes of all of the elements of the leptonic mixing matrix are then predicted in terms of the single additional parameter  $M$ . The successful ansatze predict a tau neutrino mass in the relevant range for the dark matter problem and structure formation, muon and electron neutrinos consistent with the MSW solution to the solar neutrino problem, and tau-muon neutrino mixing at a level which should soon be observed by the CHORUS and NOMAD experiments.

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# 1 Introduction

The Mikheyev-Smirnov-Wolfenstein (MSW) mechanism [1, 2, 3], which has been proposed to account for the observed deficits in neutrino flux from the Sun when compared to standard solar model (SSM) predictions, implies that the electron neutrino  $\nu_e$  may mix with another neutrino with a mass difference and mixing angle given by [4]  $\Delta m^2 \approx 3 \times 10^{-6} - 1 \times 10^{-5} \text{ eV}^2$ , and  $\sin^2 2\theta \approx 10^{-3} - 10^{-2}$ , plus a larger angle solution. The simplest theoretical interpretation of these data is that the muon neutrino has a mass  $m_{\nu_\mu} \approx 2 - 3 \times 10^{-3} \text{ eV}$  and mixes dominantly with the electron neutrino which is much lighter. The status and prospects for confirmation of these results has been thoroughly discussed elsewhere [5]. The theoretical implications of these results has also been widely discussed in the literature, most recently within the framework of supersymmetric grand unified models (SUSY GUTs) [6] based on the idea of a see-saw mechanism [7, 8].

The see-saw mechanism assumes the neutrino mass terms are of the form

$$\left[ \overline{(\nu_L)} \quad \overline{(\nu_R)^c} \right] \begin{bmatrix} 0 & m_D/2 \\ (m_D)^T/2 & M \end{bmatrix} \begin{bmatrix} (\nu_L)^c \\ (\nu_R) \end{bmatrix} + \text{h.c.} = m_D \overline{(\nu_L)} (\nu_R) + M \overline{(\nu_R)^c} (\nu_R) + \text{h.c.}, \quad (1)$$

where  $M \gg m_D$  is the Majorana mass of the right handed neutrino. Such a mass matrix has eigenvalues  $m_1 \sim m_D^2/4M$ ,  $m_2 \sim M$ . In general,  $m_D$  and  $M$  are  $3 \times 3$  matrices in family space and in order to make predictions for neutrino masses and mixing angles, one needs to know both  $m_D$  and  $M$ .

A popular back-of-the-envelope suggestion is that

$$m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} \sim m_u^2 : m_c^2 : m_t^2 \quad (2)$$

assuming  $m_D$  to be given by the corresponding up, charm and top masses. Taking equal values of  $M \sim 10^{11} \text{ GeV}$  for the three families implies  $m_{\nu_\mu} \approx 2 - 3 \times 10^{-3} \text{ eV}$ , and  $m_{\nu_\tau} \sim 50 \text{ eV}$ , which implies that the tau neutrino may play a cosmological role as hot dark matter. However if the neutrino mixing angle is assumed to be equal to the Cabibbo angle, then one would obtain  $\sin^2 2\theta \approx 0.18$  which is not in an MSW allowed region [5].

Clearly the above estimate is too simplistic: it ignores important theoretical effects due to group theoretical Clebsch coefficients, and renormalisation group (RG) running which will be important in any realistic calculation. However the problem is not just calculational it is conceptual, since the problem of neutrino masses and mixing angles is linked to the long standing problem of quark and charged lepton masses and quark mixing angles. Furthermore, the above estimate assumes a relationship between quark and lepton masses which may or may not be realised in any given theory. Finally the above estimate makes a simple assumption about the Majorana neutrino mass matrix, without which it is impossible to make any progress at all. Faced with these issues theorists have resorted to as many SUSY GUT models and approaches as there are theoretical papers on the subject, and no common consensus has yet emerged.

However it is interesting that several such models [9, 10, 11, 12] predict  $\nu_\mu - \nu_\tau$  mixing at levels which are within the range of sensitivity of the CHORUS [13], NOMAD [14] and P803 [15] experiments, which are ideally suited to  $m_{\nu\tau} \geq$  a few eV. The present limit on such mixings of  $\sin^2 2\theta \leq 5 \times 10^{-3}$  will be pushed down by over an order of magnitude, providing ample scope for an astonishing experimental discovery.

Given the existence of neutrino masses, the first deduction we could make is that the standard model is ruled out! This would also apply to those string models which give birth to the SUSY standard model at the Planck scale. Minimal SU(5) supergravity models would also be ruled out<sup>2</sup> A minimum requirement for a see-saw mechanism of the type which we are assuming is the existence of right-handed neutrino fields  $\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$ , which are singlets under the standard model gauge group. Having introduced such fields, it then becomes theoretically possible to gauge the symmetry  $B-L$ , where  $B$  is baryon number and  $L$  is lepton number, since  $B-L$  is now anomaly-free. This is theoretically very attractive since gauge symmetries are preferable to accidental global symmetries, and we find this irresistible. Having gauged  $B-L$ , it is natural to assume further unification, such as the SO(10) gauge group which contains B-L [10].

However, despite its popularity in the literature, there are serious disadvantages to choosing SO(10) which one should be aware of. It turns out that, in order to construct a right-handed neutrino Majorana matrix, one requires a large Higgs representation such as a 126 dimensional representation of SO(10). Furthermore, the scale of right-handed neutrino masses will be set by the grand unified scale  $M_X \approx 10^{16}$  GeV which is much larger than the desirable scale  $M \sim 10^{11}$  GeV encountered earlier. In order to rectify this one must rely on either Clebsch factors [10] or introduce an intermediate scale of B-L symmetry breaking into the non-SUSY model[16]. In fact in the non-SUSY version it is possible to generate suitable Majorana masses without such large Higgs representations by a two-loop mechanism [17], but in SUSY GUTs this mechanism fails because they are suppressed by the effects of the non-renormalisation theorem. In addition SO(10) is subject to the Higgs doublet-triplet splitting problem, which has no definitive solution yet within the context of string theory. Of course there are viable solutions to the doublet-triplet splitting problem available in the literature [18] but in the model we shall discuss this problem never arises in the first place, and we regard this as an attractive feature of our scheme.

There are alternatives to SO(10) in which right-handed neutrinos are mandatory and the see-saw mechanism may be exploited. For example consider two gauge groups  $SU(5) \otimes U(1)$  (commonly referred to as flipped SU(5)) [19] and  $SU(4) \otimes SU(2)^2$  (which we shall refer to as 422) [20, 21, 22, 23, 24]. Both these models are “string friendly” in the sense that they do not involve large Higgs representations, and either do not suffer from a doublet-triplet splitting problem (in the case of the 422 model) or its resolution is trivial (in the case of flipped SU(5).) Neutrino masses have been

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<sup>2</sup>Such models can easily be supplemented by non-renormalisable operators capable of giving the left-handed neutrino a Majorana mass, without the introduction of right-handed neutrinos. In this paper we shall focus only on models which exploit the see-saw mechanism, however.

studied in both models [9, 25]. In both models the neutrino Majorana matrix can be generated from higher dimensional operators, leading to a natural suppression for the scale  $M \sim M_X^2/M_{Planck}$ . However in both cases this simple picture is complicated by further singlet fields (i.e. singlets of the full gauge symmetry) which can mix with the right-handed neutrinos leading to a more complicated neutrino mass matrix. In the presence of singlets the physical neutrino masses are of order  $m_D^2\mu/M_X^2$ , where  $\mu$  is the mass scale of the singlets. Clearly if it is assumed that  $\mu$  is of order the electroweak scale then this will result in ultra-small neutrino masses. However if  $\mu \approx M_{Planck}$  then interesting neutrino masses result.

Recently we have performed a fermion mass operator analysis on the SUSY 422 model [23]. According to this analysis the entire spectrum of quark and charged lepton masses and quark mixing angles can be reproduced by a suitable set of operators, assuming renormalisable operators for the third family only and non-renormalisable operators for the lighter families. In this model the gauged B-L generator is easily identified as one of the diagonal SU(4) generators, and the leptons can be regarded as a fourth quark colour[20]. Indeed, many of the predictions valid in SO(10), such as third family Yukawa unification, and intimate relationships between quark and lepton masses via Clebsch coefficients are valid in the 422 model. For example relations analogous to those in Eq.2 may be valid in the 422 model but modified by Clebsch and RG considerations. Analogous relations would certainly not be valid in flipped SU(5). Although neutrino masses were not considered in this analysis, the Dirac part of the neutrino mass matrix is automatically predicted without further assumption. In order to obtain the physical neutrino masses and mixing angles we need to supplement the theory with the right-handed neutrino Majorana mass matrix.

In the present paper we shall extend the results of the previous operator analysis to include neutrinos, by including a very simple ansatz for the right-handed neutrino Majorana mass matrix. In the face of complete ignorance about the Majorana matrix, we shall resort to the simplest possible assumption that can be made, namely that of universal Majorana neutrino masses. We shall ignore singlet mixing completely (which is equivalent to ignoring the singlets completely) and assume that the right-handed neutrino Majorana matrix is equal to the three-dimensional unit matrix multiplied by an overall scale factor  $M$ . This is a big assumption, but at least it has the virtue of simplicity, and leads to a neutrino spectrum which is physically very interesting. To be specific, we shall find cases where the neutrino spectrum is of the following form:

$$\begin{aligned}
m_{\nu_e} \ll m_{\nu_\mu} &\approx 2 - 3 \times 10^{-3} \text{ eV} \\
\sin^2 2\theta_{e\mu} &\approx 10^{-3} - 10^{-2} \\
m_{\nu_\tau} &\geq 8 \text{ eV} \\
\sin^2 2\theta_{\mu\tau} &\approx 2 - 5 \times 10^{-3}.
\end{aligned} \tag{3}$$

This spectrum involves an inputted MSW value of the muon neutrino mass in the above range in order to set the mass parameter  $M$ . Having fixed  $M$  all the other parameters listed above are true predictions. Thus the value of the MSW mixing

angle, which is clearly in the correct range, is a genuine prediction of the scheme. We also predict the tau neutrino mass to be in the cosmologically interesting range. Finally our model predicts that CHORUS, NOMAD and P803 will all see  $\nu_\mu \rightarrow \nu_\tau$  oscillations in the near future.

The layout of the remainder of this paper is as follows: in section 2, the 422 model is introduced, specifically the superfield gauge irreducible representations and the superpotential. In section 3, the non-renormalisable operators employed to provide masses and mixings for the lighter fermions in ref.[23] are reviewed and a summary of the ansatz that gave successful predictions in the charged fermion sector is given. In section 4, the renormalisation group procedure is extended to include neutrino Yukawa couplings and the additional heavy mass scale of the right handed tau neutrino threshold. We discuss the full three family implementation of the see-saw mechanism, including complex phases in section 5. In section 6, the diagonalisation of the charged fermion and neutrino Yukawa couplings is discussed and the resulting relations between fermion masses and mixing angles are displayed. The predictive ansatz are filtered and constrained by the MSW effect, the cosmological bound and the E531 data in section 7. Finally, the prospects for the model and a summary of the neutrino mass and mixing prediction is presented in the conclusion, section 8.

## 2 The Model

Here we briefly summarise the parts of the model which are relevant for our analysis. For a more complete discussion see [21]. The gauge group is,

$$\text{SU}(4) \otimes \text{SU}(2)_L \otimes \text{SU}(2)_R. \quad (4)$$

The left-handed quarks and leptons are accommodated in the following representations,

$$F^{i\alpha a} = (4, 2, 1) = \begin{pmatrix} u^R & u^B & u^G & \nu \\ d^R & d^B & d^G & e^- \end{pmatrix}^i \quad (5)$$

$$\bar{F}_{x\alpha}^i = (\bar{4}, 1, \bar{2}) = \begin{pmatrix} \bar{d}^R & \bar{d}^B & \bar{d}^G & e^+ \\ \bar{u}^R & \bar{u}^B & \bar{u}^G & \bar{\nu} \end{pmatrix}^i \quad (6)$$

where  $\alpha = 1 \dots 4$  is an  $\text{SU}(4)$  index,  $a, x = 1, 2$  are  $\text{SU}(2)_{L,R}$  indices, and  $i = 1 \dots 3$  is a family index. The Higgs fields are contained in the following representations,

$$h_a^x = (1, \bar{2}, 2) = \begin{pmatrix} h_2^+ & h_1^0 \\ h_2^0 & h_1^- \end{pmatrix} \quad (7)$$

(where  $h_1$  and  $h_2$  are the low energy Higgs superfields associated with the MSSM.) The two heavy Higgs representations are

$$H^{\alpha b} = (4, 1, 2) = \begin{pmatrix} u_H^R & u_H^B & u_H^G & \nu_H \\ d_H^R & d_H^B & d_H^G & e_H^- \end{pmatrix} \quad (8)$$

and

$$\bar{H}_{\alpha x} = (\bar{4}, 1, \bar{2}) = \begin{pmatrix} \bar{d}_H^R & \bar{d}_H^B & \bar{d}_H^G & e_H^+ \\ \bar{u}_H^R & \bar{u}_H^B & \bar{u}_H^G & \bar{\nu}_H \end{pmatrix}. \quad (9)$$

The Higgs fields are assumed to develop VEVs,

$$\langle H \rangle = \langle \nu_H \rangle \sim M_X, \quad \langle \bar{H} \rangle = \langle \bar{\nu}_H \rangle \sim M_X \quad (10)$$

leading to the symmetry breaking at  $M_X$

$$\text{SU}(4) \otimes \text{SU}(2)_L \otimes \text{SU}(2)_R \longrightarrow \text{SU}(3)_C \otimes \text{SU}(2)_L \otimes \text{U}(1)_Y \quad (11)$$

in the usual notation. Under the symmetry breaking in Eq.11, the Higgs field  $h$  in Eq.7 splits into two Higgs doublets  $h_1, h_2$  whose neutral components subsequently develop weak scale VEVs,

$$\langle h_1^0 \rangle = v_1, \quad \langle h_2^0 \rangle = v_2 \quad (12)$$

with  $\tan \beta \equiv v_2/v_1$ .

In addition to the Higgs fields in Eqs. 8,9 the model also involves an  $\text{SU}(4)$  sextet field  $D = (6, 1, 1)$ . The superpotential of the model is a simplified version<sup>3</sup> of the one in ref.[21]:

$$W = \lambda_1^{ij} F_i \bar{F}_j h + \lambda_2 H H D + \lambda_3 \bar{H} \bar{H} D + \mu h h \quad (13)$$

Note that this is not the most general superpotential invariant under the gauge symmetry. Additional terms not included in Eq.13 may be forbidden by imposing suitable discrete symmetries, the details of which need not concern us here. The  $D$  field does not develop a VEV but the terms in Eq.13  $H H D$  and  $\bar{H} \bar{H} D$  combine the colour triplet parts of  $H, \bar{H}$  and  $D$  into acceptable GUT-scale mass terms [21]. When the  $H$  fields attain their VEVs at  $M_X \sim 10^{16}$  GeV, the superpotential of Eq.13 reduces to the MSSM. Note that the last term in Eq.13 is proportional to the dimensionful parameter  $\mu$ . This parameter could be generated by a gauge singlet field that attains a VEV of order the weak scale, as in the Next-to-Minimal Supersymmetric Standard Model (NMSSM) [26, 27]. In the NMSSM, an extra term must be added to Eq.13 but this is not expected to significantly change any of the results in this paper [28]. Below  $M_X$  the part of the superpotential involving matter superfields is just

$$W = \lambda_U^{ij} Q_i \bar{U}_j h_2 + \lambda_D^{ij} Q_i \bar{D}_j h_1 + \lambda_E^{ij} L_i \bar{E}_j h_1 + \lambda_N^{ij} L_i N_j h_2 + \dots \quad (14)$$

where  $N_i$  are the superfields associated with the right-handed neutrinos. The Yukawa couplings in Eq.14 satisfy the boundary conditions

$$\lambda_1^{ij}(M_X) \equiv \lambda_U^{ij}(M_X) = \lambda_D^{ij}(M_X) = \lambda_E^{ij}(M_X) = \lambda_N^{ij}(M_X). \quad (15)$$

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<sup>3</sup>Without Higgs singlet fields.

### 3 Operator Analysis

In this section, in order to make this paper self-contained, we briefly review the results of the operator analysis of ref.[23], extending the resulting tables of Clebsch coefficients to include the neutrinos.

The boundary conditions listed in Eq.15 lead to unacceptable mass relations for the light two families. Also, the large family hierarchy in the Yukawa couplings appears to be unnatural since one would naively expect the dimensionless couplings all to be of the same order. This leads us to the conclusion that the  $\lambda_1^{ij}$  in Eq.13 may not originate from the usual renormalisable tree level dimensionless coupling. We allow a renormalisable Yukawa coupling in the 33 term only and generate the rest of the effective Yukawa couplings by non-renormalisable operators that are suppressed by some higher mass scale. This suppression provides an explanation for the observed fermion mass hierarchy.

We shall restrict ourselves to all possible non-renormalisable operators which can be constructed from different group theoretical contractions of the fields:

$$O_{ij} \sim (F_i \bar{F}_j) h \left( \frac{H \bar{H}}{M_S^2} \right)^n + \text{H.c.} \quad (16)$$

where we have used the fields  $H, \bar{H}$  in Eqs.8,9 and  $M_S$  is the large scale  $M_S > M_X$ . The idea is that when  $H, \bar{H}$  develop their VEVs such operators will become effective Yukawa couplings of the form  $h F \bar{F}$  with a small coefficient of order  $M_X^2/M_S^2$ . We shall only consider up to  $n = 2$  operators here, since as we shall see even at this level there are a wealth of possible operators that are encountered.

We shall assume that the Yukawa matrices at  $M_X$  are all of the form

$$Y^{U,D,E,N} = \begin{pmatrix} O(\epsilon^2) & O(\epsilon^2) & 0 \\ O(\epsilon^2) & O(\epsilon) & O(\epsilon) \\ 0 & O(\epsilon) & O(1) \end{pmatrix}, \quad (17)$$

where  $\epsilon \ll 1$  and some of the elements may have approximate or exact texture zeroes in them. Eq.17 allows us to consider the lower 2 by 2 block of the Yukawa matrices first. In diagonalising the lower 2 by 2 block separately, we introduce corrections of order  $\epsilon^2$  and so the procedure is consistent to first order in  $\epsilon$ . In a previous paper [23], we found several maximally predictive ansatze that were constructed out of the operators whose Clebsch coefficients are listed in table 1 for the  $n = 1$  operators. The explicit operators in component form are listed in the appendices of ref.[23]. These  $n = 1$  operators were used in the lower right hand block of the Yukawa matrices. We label these successful lower 2 by 2 ansatze  $A_i$ :

$$A_1 = \begin{bmatrix} O_{22}^D - O_{22}^C & 0 \\ O_{32}^B & O_{33} \end{bmatrix} \quad (18)$$

$$A_2 = \begin{bmatrix} 0 & O_{23}^A - O_{23}^B \\ O_{32}^D & O_{33} \end{bmatrix} \quad (19)$$

	$QUh_2$	$QDh_1$	$LEh_1$	$LNh_2$
$O^A$	1	1	1	1
$O^B$	1	-1	-1	1
$O^C$	1	1	-3	-3
$O^D$	1	-1	3	-3
$O^G$	0	1	2	0
$O^H$	2	1	2	4
$O^K$	1	0	0	3/4
$O^M$	0	1	1	0
$O^N$	1	0	0	0
$O^O$	1	0	0	2

Table 1: When the Higgs fields develop their VEVs at  $M_X$ , the  $n = 1$  operators utilised lead to the effective Yukawa couplings with Clebsch coefficients as shown.

$$A_3 = \begin{bmatrix} 0 & O_{23}^C - O_{23}^D \\ O_{32}^B & O_{33} \end{bmatrix} \quad (20)$$

$$A_4 = \begin{bmatrix} 0 & O_{23}^C \\ O_{32}^A - O_{32}^B & O_{33} \end{bmatrix} \quad (21)$$

$$A_5 = \begin{bmatrix} 0 & O_{23}^A \\ O_{32}^C - O_{32}^D & O_{33} \end{bmatrix} \quad (22)$$

$$A_6 = \begin{bmatrix} O_{22}^K & O_{23}^C \\ O_{32}^M & O_{33} \end{bmatrix} \quad (23)$$

$$A_7 = \begin{bmatrix} O_{22}^K & O_{23}^G \\ O_{32}^G & O_{33} \end{bmatrix} \quad (24)$$

$$A_8 = \begin{bmatrix} 0 & O_{23}^H \\ O_{32}^G - O_{32}^K & O_{33} \end{bmatrix}. \quad (25)$$

Note that replacing  $O^K$  with  $O^N$  or  $O^O$  in  $A_{6-8}$  would yield the same results in the charged fermion sector. The choice of this operator is however crucial when one considers neutrino masses and so we denote the choice  $A_i^K, A_i^N$  or  $A_i^O$ , where  $i = 6, 7, 8$ .

From the above ansatze in Eqs.18-25, the ratio of muon to strange Yukawa couplings at  $M_X$  is found to be:

$$\left( \frac{\lambda_\mu}{\lambda_s} \right)_{M_X} \equiv l. \quad (26)$$

where  $l$  is a ratio of Clebsch coefficients, predicted to be  $l = 3$ , as in the Georgi-Jarlskog (GJ) [29] ansatz or  $l = 4$  (a new prediction). Ansätze  $A_{1-6}$  predict  $l = 3$  in Eq.26 and ansätze  $A_{7,8}$  predict  $l = 4$ .

The effective Yukawa couplings generated by the  $n = 2$  operators have Clebsch coefficients associated with them, as displayed in Table 2. Similarly, we now list the



	$QUH_2$	$Q\bar{D}H_1$	$LEH_1$	$LNH_2$
$O^{Ad}$	1	3	9/4	3/4
$O^{Dd}$	1	3	3	1
$O^{Md}$	1	3	6	2
$O^1$	0	1	1	0
$O^2$	0	1	3/4	0
$O^3$	0	1	2	0

Table 2:  $n = 2$  operators utilised in the upper 2 by 2 ansatze  $B_i$ .

$n = 2$  operators that are used to account for  $|V_{us}|, m_u, m_d, m_e$  and  $|V_{ub}|$ . The possible ansatze in the down sector that account for correct  $|V_{us}|, m_u, m_d, m_e$  and  $|V_{ub}|$  and *also* generate CP violation are:

$$B_1 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^1 \\ O_{21}^{Ad} & X \end{bmatrix} \quad (27)$$

$$B_2 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^2 \\ O_{21}^{Ad} & X \end{bmatrix} \quad (28)$$

$$B_3 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^3 \\ O_{21}^{Ad} & X \end{bmatrix} \quad (29)$$

$$B_4 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^1 \\ O_{21}^{Dd} & X \end{bmatrix} \quad (30)$$

$$B_5 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^2 \\ O_{21}^{Dd} & X \end{bmatrix} \quad (31)$$

$$B_6 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^3 \\ O_{21}^{Dd} & X \end{bmatrix} \quad (32)$$

$$B_7 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^1 \\ O_{21}^{Md} & X \end{bmatrix} \quad (33)$$

$$B_8 = \begin{bmatrix} 0 & O_{12}^{n=3} + O_{12}^2 \\ O_{21}^{Md} & X \end{bmatrix}. \quad (34)$$

$X$  stands for the operator(s) in the 22 position as given earlier. Each of the successful ansatze, consisting of one of the  $A_i$  combined with one of the  $B_j$ , gives a prediction for the down Yukawa coupling at  $M_X$  in terms of the electron Yukawa coupling:

$$\left( \frac{\lambda_d}{\lambda_e} \right)_{M_X} \equiv k, \quad (35)$$

where  $k = 3$  is the GJ prediction of  $m_d$ . Other viable possibilities found in our analysis are  $k = 2, 4, \frac{8}{3}, \frac{16}{3}$  as shown in Table 3. When the CKM matrix from the ansatze  $A_i, B_j$  is calculated, a prediction of  $|V_{ub}|$  in terms of an unknown complex

$k$	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$	$B_7$	$B_8$
$A_{1-6}$	4	16/3	2	3	4	(3/2)	(3/2)	2
$A_{7,8}$	16/3	(64/9)	8/3	4	16/3	2	2	8/3

Table 3:  $k$  values predicted by the ansatze  $A_1$  to  $A_8$  when combined with  $B_1$  to  $B_8$ . Note that the bracketed entries predict  $m_d(1 \text{ GeV})$  to be outside the empirical range.

phase  $\phi$  is found:

$$|V_{ub}(m_t)| \sim \frac{|V_{us}(m_t)||V_{cb}(m_t)|}{\sqrt{1 + \left(\frac{3\lambda_c(M_X)}{\lambda_s(M_X)}\right)^2 - 2 \cos \phi(M_X) \frac{3\lambda_c(M_X)}{\lambda_s(M_X)}}}. \quad (36)$$

More details of this analysis can be found in ref.[23].

## 4 Generalisation to Include Neutrino Masses

This analysis of charged fermion masses does not include neutrinos, even though the model predicts Dirac neutrino masses to be of order the mass of the up quarks. If this was the case, direct upper neutrino mass bounds would be violated and so some mechanism must be employed to suppress them significantly. In this paper, we assume that the neutrino masses are suppressed via the popular see saw mechanism, as in Eq.1. Since the Dirac masses  $m_D$  of  $\nu_e$  and  $\nu_\mu$  are several orders of magnitude smaller than that of  $\nu_\tau$ , it is a good approximation to consider the third family alone and drop smaller Yukawa couplings, as we did for the charged fermions.

As shown in ref. [30], the tau neutrino Yukawa coupling enters the system of renormalisation group equations (RGEs) of the third family Yukawa couplings. However, the effects of this coupling on the predictions of triple Yukawa unification [10] are negligible compared to the effects of the experimental uncertainties on  $m_b$  and  $\alpha_S(M_Z)$ . Thus the numerical predictions of  $m_d, m_s, m_b, \tan \beta, m_t$  and  $|V_{ub}|$  made in ref.[23] still approximately hold once the neutrinos have been taken into account. At the scale  $M$ , the right handed tau neutrino decouples from the effective field theory and so if one assumes that  $M \sim 10^{16} \text{ GeV}$ , the predictions in ref.[23] hold completely. Such a high value of the Majorana mass scale would mean very light neutrinos of masses  $< O(10^{-3}) \text{ eV}$ , which could not account for any of the solar neutrino, dark matter or structure formation problems in cosmology. On the contrary, Majorana masses of order  $10^{10} - 10^{12} \text{ GeV}$  can address these problems [30] and it is in this region of the parameter space that we are going to analyse the ansatze  $A_i, B_j$  to obtain neutrino masses and leptonic mixing angles.

The superpotential in Eq.13 was used to calculate the running of the Yukawa couplings in the MSSM and the  $\overline{MS}$  renormalisation scheme, using an analysis of

general superpotentials performed by Martin and Vaughn [31]:

$$\begin{aligned}
\frac{\partial g_i}{\partial t} &= \frac{b_i g_i^3}{16\pi^2} \\
\frac{\partial Y^U}{\partial t} &= \frac{Y^U}{16\pi^2} \left[ \text{Tr} \left( 3Y^U Y^{U\dagger} + Y^{N\dagger} Y^N \right) + 3Y^{U\dagger} Y^U + Y^{D\dagger} Y^D - \left( \frac{13}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
\frac{\partial Y^D}{\partial t} &= \frac{Y^D}{16\pi^2} \left[ \text{Tr} \left( 3Y^D Y^{D\dagger} + Y^{E\dagger} Y^E \right) + Y^{U\dagger} Y^U + 3Y^{D\dagger} Y^D - \left( \frac{7}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
\frac{\partial Y^E}{\partial t} &= \frac{Y^E}{16\pi^2} \left[ \text{Tr} \left( 3Y^D Y^{D\dagger} + Y^{E\dagger} Y^E \right) + 3Y^{E\dagger} Y^E + Y^{N\dagger} Y^N - \left( \frac{9}{5} g_1^2 + 3g_2^2 \right) \right] \\
\frac{\partial Y^N}{\partial t} &= \frac{Y^N}{16\pi^2} \left[ \text{Tr} \left( 3Y^{U\dagger} Y^U + Y^{N\dagger} Y^N \right) + 3Y^{N\dagger} Y^N + Y^{E\dagger} Y^E - \left( \frac{3}{5} g_1^2 + 3g_2^2 \right) \right], \quad (37)
\end{aligned}$$

where  $b_i = (33/5, 1, -3)$ ,  $t = \ln \mu$  and  $\mu$  is the  $\overline{MS}$  renormalisation scale. Once the small couplings have been dropped, Eqs.37 reduce to the RGEs derived in [32]:

$$\begin{aligned}
16\pi^2 \frac{\partial g_i}{\partial t} &= b_i g_i^3 \\
16\pi^2 \frac{\partial \lambda_t}{\partial t} &= \lambda_t \left[ 6\lambda_t^2 + \lambda_b^2 + \theta_R \lambda_{\nu_\tau}^2 - \left( \frac{13}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_b}{\partial t} &= \lambda_b \left[ 6\lambda_b^2 + \lambda_\tau^2 + \lambda_t^2 - \left( \frac{7}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_\tau}{\partial t} &= \lambda_\tau \left[ 4\lambda_\tau^2 + 3\lambda_b^2 + \theta_R \lambda_{\nu_\tau}^2 - \left( \frac{9}{5} g_1^2 + 3g_2^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_{\nu_\tau}}{\partial t} &= \lambda_{\nu_\tau} \left[ 4\theta_R \lambda_{\nu_\tau}^2 + 3\lambda_t^2 + \lambda_\tau^2 - \left( \frac{3}{5} g_1^2 + 3g_2^2 \right) \right], \quad (38)
\end{aligned}$$

where  $\theta_R \equiv \theta(t - \ln M)$  takes into account the large mass suppression of the right-handed neutrino loops at scales  $\mu < M$ . Thus we integrate out loops involving right-handed neutrinos at  $M$ , but retain the Dirac Yukawa coupling  $\lambda_{\nu_\tau}$  which describes the coupling of left to right-handed neutrinos. The running procedure to determine the low energy masses is then to run down the (Dirac) neutrino Yukawa coupling  $\lambda_{\nu_\tau}$  from  $M_X$  to low-energies using the above RGEs.

In dealing with the first and second families we have to confront the problem that the Yukawa matrices are not diagonal. As discussed widely elsewhere [10, 33], it is most convenient to diagonalise the Yukawa matrices at  $M_X$  before running them down to  $m_t$ . It is then possible to obtain RGEs for both the *diagonal* Yukawa couplings  $\lambda_{u,c,t}$ ,  $\lambda_{d,s,b}$ ,  $\lambda_{e,\mu,\tau}$  and the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements  $|V_{ij}|^4$  (ref.[28, 34, 35, 36, 37]). At one-loop these RGEs can be numerically integrated so that the low energy physical couplings have a simple scaling behaviour

$$\left( \frac{\lambda_{u,c}}{\lambda_t} \right)_{m_t} = \left( \frac{\lambda_{u,c}}{\lambda_t} \right)_{M_X} e^{3I_t + I_b}$$

---

<sup>4</sup>The empirical values of  $|V_{ij}|$  were taken to be at  $m_t$  instead of  $M_Z$ , introducing an error whose magnitude is always less than 1 percent for our analysis.

$$\begin{aligned}
\left(\frac{\lambda_{d,s}}{\lambda_b}\right)_{m_t} &= \left(\frac{\lambda_{d,s}}{\lambda_b}\right)_{M_X} e^{3I_b+I_t} \\
\left(\frac{\lambda_{e,\mu}}{\lambda_\tau}\right)_{m_t} &= \left(\frac{\lambda_{e,\mu}}{\lambda_\tau}\right)_{M_X} e^{3I_\tau+I_{\nu_\tau}} \\
\left(\frac{\lambda_{\nu_e,\nu_\mu}}{\lambda_{\nu_\tau}}\right)_{m_t} &= \left(\frac{\lambda_{\nu_e,\nu_\mu}}{\lambda_{\nu_\tau}}\right)_{M_X} e^{I_\tau+3I_{\nu_\tau}} \\
\frac{|V_{cb}|_{M_X}}{|V_{cb}|_{m_t}} &= e^{I_b+I_t} \\
\frac{|V_{\mu\tau}|_{M_X}}{|V_{\mu\tau}|_{m_t}} &= e^{I_\tau+I_{\nu_\tau}},
\end{aligned} \tag{39}$$

with identical scaling behaviour to  $V_{cb}$  of  $V_{ub}$ ,  $V_{ts}$ ,  $V_{td}$ . The leptonic mixing elements  $V_{e\tau}$ ,  $V_{\tau e}$ ,  $V_{\tau\mu}$  scale identically to  $V_{\mu\tau}$ . The  $I$  integrals are defined as

$$I_i \equiv \int_{\Lambda_i}^{\ln M_X} \left(\frac{\lambda_i(t)}{4\pi}\right)^2 dt, \tag{40}$$

where  $\Lambda_{\nu_\tau} = M$ ,  $\Lambda_{\tau,b,t} = m_t$  and  $t = \ln \mu$ ,  $\mu$  being the  $\overline{MS}$  scale. To a consistent level of approximation  $V_{e\mu}$ ,  $V_{ee}$ ,  $V_{\mu\mu}$ ,  $V_{\mu e}$ ,  $V_{\tau\tau}$ ,  $V_{us}$ ,  $V_{ud}$ ,  $V_{cs}$ ,  $V_{cd}$ ,  $V_{tb}$ ,  $\lambda_{\nu_e}/\lambda_{\nu_\mu}$ ,  $\lambda_u/\lambda_c$ ,  $\lambda_d/\lambda_s$  and  $\lambda_e/\lambda_\mu$  are RG invariant. The CP violating quantity  $J$  scales as  $V_{cb}^2$ .

## 5 Implementing the See-Saw Mechanism

For simplicity, and to make the scheme as predictive as possible, we assume the Majorana matrix  $M$  to be proportional to the unit matrix in family space.  $M$  may be generated by non-renormalisable terms of the form

$$\lambda \bar{F}_{\alpha x}^j \bar{F}_{\beta y}^j \frac{H^{\alpha x} H^{\beta y}}{M_{S_1}} \left( \frac{H^{\gamma z} \bar{H}_{\gamma z}}{M_{S_2}} \right)^m, \tag{41}$$

where  $M_{S_i} > M_X$ ,  $i = 1, 2$  are higher scales<sup>5</sup>. We do not explicitly specify the exact operator responsible for  $M$ , but merely use a numerical value of  $10^{10} - 10^{12}$  GeV as a starting point to solve some of the problems associated with neutrino masses.

The diagonalisation procedure to transform to the mass basis of the neutrinos seems more complicated than a similar calculation on the charged fermions because of the additional feature of the Majorana mass matrix. We now show how the neutrino masses may be diagonalised in our scheme. We take the mass matrix from Eq.1, generalise to the three family case and diagonalise the Dirac masses by unitary transformations upon the neutrino fields at the unification scale  $M_X$ :

$$\left[ \overline{(\nu_L)_i} \overline{(\nu_R)_i^c} \right] (V^+ V)^T \begin{bmatrix} 0 & m_D/2 \\ m_D^T/2 & M \end{bmatrix} V^+ V \begin{bmatrix} (\nu_L)_i^c \\ (\nu_R)_i \end{bmatrix} + \text{H.c.}, \tag{42}$$

---

<sup>5</sup>For example the compactification scale, the string scale or the Planck scale.

where each 6 by 6 matrix has been split up into four 3 by 3 submatrices and

$$V \equiv \begin{bmatrix} V_{N_L}^* & 0 \\ 0 & V_{N_R} \end{bmatrix}. \quad (43)$$

We redefine the neutrino superfields as  $(\nu_L)_i \rightarrow V_{N_L}^{ij}(\nu_L)_j$  and  $(\nu_R)_i \rightarrow V_{N_R}^{ij}(\nu_R)_j$  in a basis where  $m_D \rightarrow V_{N_L} m_D V_{N_R}^+$  is diagonal. We denote the diagonalised entries of  $m_D$  as  $m_i, (i = 1 \dots 3)$ . Note that the Majorana matrix  $M \rightarrow V_{N_R}^* M V_{N_R}^+ = M \text{diag}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau})$  (with the last equality following from the assumption that  $M$  is proportional to the unit matrix) now involves complex phases in general. The  $(\nu_R)_i$  superfields are redefined (to a primed basis) to absorb the complex phases present in the Majorana mass terms

$$M \sum_{i=e,\mu,\tau} e^{i\phi_i} \overline{(\nu_R)^c}_i (\nu_R)_i = M \sum_{i=e,\mu,\tau} \overline{(\nu'_R)^c}_i (\nu'_R)_i \quad (44)$$

where

$$\nu'_{e,\mu,\tau R} = e^{i\phi_{e,\mu,\tau}/2} \nu_{e,\mu,\tau R}, \quad (45)$$

and has the effect of making the neutrino Dirac entries carry phases, which may then be absorbed into a redefinition of the left handed neutrino phases

$$(\nu_L)'_i = e^{-i\phi_i/2} (\nu_L)_i \quad (46)$$

leaving all of the neutrino masses real. The transformation in Eq.46 will produce phases in the leptonic CKM mixing matrix  $V_{LEP}$ , which will be defined in the next section. However it will turn out that these phases contribute to the overall phase of individual matrix elements  $V_{LEPij}$  and because our predictions are only in terms of the moduli of these angles, they do not enter our calculations. In fact it is easy to see that, because of our assumption that the Majorana matrix is proportional to the unit matrix,  $|V_{LEP}|_{ij}$  may be calculated just from the Dirac parts of the leptonic matrices ignoring the see-saw mechanism completely.

Thus the strategy is to diagonalise the Dirac Yukawa matrices at  $M_X$ , and obtain the diagonal Yukawa couplings and quark and lepton CKM matrix elements at the scale  $M_X$ . These parameters are then run to low energies using the procedure outlined in section 4, i.e. taking into account the effects of the right handed tau neutrino. The Majorana masses do not run to one loop. At low energy ( $\mu = M_W$ ), in order to obtain the light neutrino basis, we do another transformation upon the neutrino mass matrix that performs the see saw action. To first order in the small see-saw angle matrix  $\Theta_i = \text{diag}(\theta_1, \theta_2, \theta_3)$ ,

$$\left[ \overline{(\nu'_L)_i} \overline{(\nu'_R)^c}_i \right] U U^T \begin{bmatrix} 0 & m_i/2 \\ m_i/2 & M \end{bmatrix} U U^T \begin{bmatrix} (\nu'_L)_i^c \\ (\nu'_R)_i \end{bmatrix} + \text{H.c.} \quad (47)$$

where

$$U \equiv \begin{bmatrix} 1 & -\Theta_i \\ \Theta_i & 1 \end{bmatrix}. \quad (48)$$

The mass matrix in Eq.47 has three light eigenvalues (the physical neutrinos) of magnitude  $m_{\nu_i} \sim -m_i^2/4M$  and three heavy eigenvalues  $\sim M$ . The neutrino mixing angles are  $\Theta_i \sim \text{diag}(-m_1/2M, -m_2/2M, -m_3/2M)$  so that the whole 6 by 6 neutrino mass matrix is now diagonal.

## 6 Predictions of Leptonic CKM Matrix Elements

The successful ansatzes consist of any of the lower 2 by 2 blocks  $A_i$  combined with any of the upper 2 by 2 blocks  $B_i$ , subject to the restrictions shown in Table 3. For example let us consider  $A_1$  in the lower 2 by 2 block combined with any of the  $B_i$  in the upper 2 by 2 block, focusing particularly on  $A_1$  combined with  $B_1$ . Just above  $M_X$ , before the  $H, \bar{H}$  fields develop VEVs, we have the operators

$$\begin{bmatrix} 0 & O_{12}^1 + O_{12}^{n=3} & 0 \\ O_{21}^{Ad} & O_{22}^D - O_{22}^C & 0 \\ 0 & O_{32}^B & O_{33} \end{bmatrix}, \quad (49)$$

which implies that at  $M_X$  the Yukawa matrices are of the form

$$Y^{U,D,E,N} = \begin{bmatrix} 0 & H_{12}x_{12}^{U,D,E,N}e^{i\phi_{12}} + H_{12}'x_{12}'^{U,D,E,N}e^{i\phi_{12}'} & 0 \\ H_{21}x_{21}^{U,D,E,N}e^{i\phi_{21}} & H_{22}x_{22}^{U,D,E,N}e^{i\phi_{22}} - H_{22}'x_{22}'^{U,D,E,N}e^{i\phi_{22}'} & 0 \\ 0 & H_{32}x_{32}^{U,D,E,N}e^{i\phi_{32}} & H_{33}e^{i\phi_{33}} \end{bmatrix}, \quad (50)$$

where we have factored out the phases of the operators and  $H_{ik}$  are the magnitudes of the coupling constant associated with  $O_{ik}$ . Note the real Clebsch coefficients<sup>6</sup>  $x_{ik}^{U,D,E,N}$  give the splittings between  $Y^{U,D,E,N}$ . We now make the transformation in the 22 element of Eq.50

$$x_{22}^{U,D,E,N}H_{22}e^{i\phi_{22}} - x_{22}'^{U,D,E,N}H_{22}'e^{i\phi_{22}'} \equiv H_{22}^{U,D,E,N}e^{i\phi_{22}^{U,D,E,N}}, \quad (51)$$

where  $H_{22}^{U,D,E,N}, \phi_{22}^{U,D,E,N}$  are real positive parameters. It follows from the Clebsch structure in Eq.72 that  $H_{22}^E = 3H_{22}^D$ ,  $H_{22}^N = 3H_{22}^U$  and  $\phi_{22}^E = \phi_{22}^D$ ,  $\phi_{22}^N = \phi_{22}^U$ . In general we shall write  $H_{22}^E = lH_{22}^D$ ,  $H_{22}^N = l_NH_{22}^U$ , where  $l = 3, l_N = 3$  in this case.

We rotate the phases of the  $F, \bar{F}$  fields as in the appendix in order to decrease the number of phases in the Yukawa matrices and hence derive real mass eigenstates eventually. Below  $M_X$ , the multiplets  $F, \bar{F}$  are no longer connected by the gauge symmetry in the effective field theory, since it is The Standard Model. We now define our notation as regards the effective field theory below  $M_X$  as follows. The effective quark Yukawa terms are written (suppressing all indices)

$$(U_R)^c Y^U Q_L h_2 + (D_R)^c Y^D Q_L h_1 + \text{H.c.} \quad (52)$$

We transform to the quark mass basis by introducing four 3 by 3 unitary matrices  $V_{U,L,R}, V_{D,L,R}$  then the Yukawa terms become

$$(U_R)^c V_{U_R}^\dagger V_{U_R} Y^U V_{U_L}^\dagger V_{U_L} Q_L h_2 + (D_R)^c V_{D_R}^\dagger V_{D_R} Y^D V_{D_L}^\dagger V_{D_L} Q_L h_1 + \text{H.c.} \quad (53)$$

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<sup>6</sup>See appendix.

where  $Y_{\text{diag}}^U = V_{U_R} Y^U V_{U_L}^\dagger$  and  $Y_{\text{diag}}^D = V_{D_R} Y^D V_{D_L}^\dagger$  are the diagonalised Yukawa matrices. With the definitions in Eqs.52,53, the CKM matrix is of the form

$$V_{CKM} \equiv V_{U_L} V_{D_L}^\dagger. \quad (54)$$

The effective lepton Yukawa terms below  $M_X$  are defined to be (suppressing all indices)

$$(\nu_R)^c Y^N L_L h_2 + (e_R)^c Y^E L_L h_1 + \text{H.c.} \quad (55)$$

We transform to the lepton dirac mass basis by introducing the four 3 by 3 unitary matrices  $V_{N_{L,R}}, V_{E_{L,R}}$  and the Yukawa terms become

$$(\nu_R)^c V_{N_R}^+ V_{N_R} Y^N V_{N_L}^+ V_{N_L} L_L h_2 + (e_R)^c V_{E_R}^+ V_{E_R} Y^E V_{E_L}^+ V_{E_L} L_L h_1 + \text{H.c.} \quad (56)$$

where  $Y_{\text{diag}}^N = V_{N_R} Y^N V_{N_L}^+$  and  $Y_{\text{diag}}^E = V_{E_R} Y^E V_{E_L}^+$  are the diagonalised lepton Yukawa matrices. The argument in section 5 shows that the assumption of a Majorana matrix proportional to 1 allows us to be only concerned with the diagonalisation of the Dirac mass matrices to calculate the magnitudes of the leptonic mixing matrix elements

$$|V_{LEP}|_{ij} \equiv |(V_{N_L})_{ik} (V_{E_L}^+)_{kj}|. \quad (57)$$

In all of the cases considered,  $x_{12}^U = 0$  and  $x_{12}^{D,E} = 0$  so that the Yukawa matrices which result from Eqs.50,51,73 are

$$\begin{aligned} Y^D &= \begin{bmatrix} 0 & H_{12} x_{12}^D & 0 \\ H_{21} x_{21}^D & H_{22}^D & 0 \\ 0 & H_{32} x_{32}^D & H_{33} \end{bmatrix} \\ Y^E &= \begin{bmatrix} 0 & H_{12} x_{12}^E & 0 \\ H_{21} x_{21}^E & l H_{22}^E & 0 \\ 0 & H_{32} x_{32}^E & H_{33} \end{bmatrix} \\ Y^U &= \begin{bmatrix} 0 & H'_{12} x_{12}^U e^{i(\phi_{12}' - \phi_{12})} & 0 \\ x_{21}^U H_{21} & H_{22}^U e^{i(\phi_{22}^U - \phi_{22}^D)} & 0 \\ 0 & H_{32} x_{32}^U & H_{33} \end{bmatrix} \\ Y^N &= \begin{bmatrix} 0 & 0 & 0 \\ x_{21}^N H_{21} & l_N H_{22}^U e^{i(\phi_{22}^U - \phi_{22}^D)} & 0 \\ 0 & H_{32} x_{32}^N & H_{33} \end{bmatrix}. \end{aligned} \quad (58)$$

Note that in Eq.58, we have assumed that the  $O_{12}^{n=3}$  operator doesn't give a Yukawa term to the neutrino for simplicity. None of the predictions change if this limit is revoked except that the electron neutrino is approximately massless rather than exactly massless. In order to diagonalise the quark Yukawa matrices, we first make  $Y^{U,N}$  real, by multiplying by phase matrices, as in the appendix.

The diagonal Yukawa couplings of the strange quark and muon obtained from Eq.75 are  $(\lambda_s)_{M_X} = H_{22}^D$  and  $(\lambda_\mu)_{M_X} = l H_{22}^D$  since the 22 eigenvalues are just the 22

elements in this case. Similarly, we use  $(\lambda_c)_{M_X} = H_{22}^U$  and  $(\lambda_{\nu_\mu})_{M_X} = l_N H_{22}^U$ . This gives us the prediction of the strange quark and neutrino Yukawa couplings

$$\lambda_s(M_X) = \lambda_\mu(M_X)/l, \quad \lambda_{\nu_\mu} = l_N \lambda_c. \quad (59)$$

$A_6$  makes no such prediction and so we discard it on the grounds that we are searching for the most predictive (and therefore testable) cases.  $A_7^N, A_8^N$  are special cases because they predict a massless muon neutrino. We consider these cases later.

The first family diagonal Yukawa couplings for the down quark and electron are related by

$$\left(\frac{\lambda_d}{\lambda_e}\right)_{M_X} = l \frac{x_{21}^D x_{12}^D}{x_{21}^E x_{12}^E}. \quad (60)$$

We identify the right hand side of Eq.60 with  $k$  in Eq.35. We now substitute the diagonalising matrices from Eqs.75 and 74 into the CKM matrix in Eq.54 to obtain

$$|V_{CKM}|_{ij} = \begin{bmatrix} c_2 c_1 e^{i\phi} + s_2 s_1 c_3 & -s_2 c_1 e^{i\phi} + s_1 c_2 c_3 & s_1 s_3 \\ -s_1 c_2 e^{i\phi} + c_1 s_2 c_3 & s_1 s_2 e^{i\phi} + c_2 c_3 c_1 & s_3 c_1 \\ -s_2 s_3 & -c_2 s_3 & c_3 \end{bmatrix}. \quad (61)$$

Substituting Eqs. 75 and 74 into the leptonic mixing matrix in Eq. 57 yields the matrix  $V_{LEP}$  form as  $V_{CKM}$ , except with all angles  $\theta_i$  replaced by their leptonic analogues  $\theta_i^l$ , defined as in the appendix.

We may start making predictions in the leptonic sector at  $M_X$  for the elements of  $|V_{LEP}|$ , in terms of those of  $|V_{CKM}|$ . Proceeding in the manner laid out in the appendix, we obtain

$$|V_{\mu\tau}(M_X)| = m |V_{cb}(M_X)| \quad (62)$$

$$|V_{e\tau}(M_X)| = n |V_{ub}(M_X)| \quad (63)$$

$$|V_{\tau e}(M_X)| = o |V_{ub}(M_X)| \frac{\lambda_c}{\lambda_\mu} \quad (64)$$

$$|V_{e\mu}(M_X)| = p \frac{|V_{ub}(M_X)|}{|V_{cb}(M_X)|} \sqrt{\left(\frac{q \lambda_c(M_X)}{\lambda_\mu(M_X)}\right)^2 - 2q \frac{\lambda_c(M_X)}{\lambda_\mu(M_X)} \cos \phi + 1}, \quad (65)$$

where  $m, n, o, p, q$  are all rational numerical Clebsch coefficients, as shown in the appendix. The uncertainty due to the arbitrary phase  $\phi$  gives a range of values for  $|V_{e\mu}|$ , even if all of the other parameters are fixed. However, it was seen in ref.[23] that the values of  $|V_{ub}|$  predicted by Eq.36 are only within the experimentally measured range for  $\phi > \pi/2$ . When making predictions, we therefore examine the endpoints of the valid range, that is  $\cos \phi = 0, -1$ .

The diagonalisation procedure in the two special cases  $A_7^N, A_8^N$  is somewhat simpler since the neutrino Yukawa matrix consists of only two non-zero entries.  $A_8^N$  is discarded because it does not make a prediction for  $|V_{\mu\tau}|$ . When  $A_7^N$  combined with any of the  $B_i$  ansatze is diagonalised, the leptonic mixing matrix is

$$V_{LEP} = \begin{bmatrix} c_2^l & -s_2^l & 0 \\ \bar{c}_4^l s_2^l & c_2^l \bar{c}_4^l & -\bar{s}_4^l \\ 0 & c_2^l \bar{s}_4^l & \bar{c}_4^l \end{bmatrix}, \quad (66)$$



$(l_N, m, n, o, p, q)$	$B_{1,2,3}$	$B_{4,5,6}$	$B_{7,8}$
$A_{1,3,4}$	$(3, 1, \frac{1}{4}, \frac{9}{4}, \frac{1}{4}, 9)$	$(3, 1, \frac{1}{3}, 3, \frac{1}{3}, 9)$	$(3, 1, \frac{2}{3}, 6, \frac{2}{3}, 9)$
$A_{2,5}$	$(3, 3, \frac{3}{4}, \frac{27}{4}, \frac{1}{4}, 9)$	$(3, 3, 1, 9, \frac{1}{3}, 9)$	$(3, 3, 2, 18, \frac{2}{3}, 9)$
$A_7^K$	$(\frac{3}{4}, 2, 2, \frac{9}{2}, 1, \frac{9}{4})$	$(\frac{3}{4}, 2, \frac{8}{3}, 6, \frac{4}{3}, \frac{9}{4})$	$(\frac{3}{4}, 2, \frac{16}{3}, 12, \frac{8}{3}, \frac{9}{4})$
$A_7^O$	$(2, 2, \frac{3}{4}, \frac{9}{2}, \frac{3}{8}, 6)$	$(2, 2, 1, 6, \frac{1}{2}, 6)$	$(2, 2, 2, 12, 1, 6)$
$A_8^K$	$(\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{27}{16}, \frac{1}{2}, \frac{9}{2})$	$(\frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{9}{4}, \frac{2}{3}, \frac{9}{2})$	$(\frac{3}{2}, \frac{3}{4}, 1, \frac{9}{2}, \frac{4}{3}, \frac{9}{2})$
$A_8^O$	$(4, 2, \frac{3}{8}, \frac{9}{2}, \frac{3}{16}, 12)$	$(4, 2, \frac{1}{2}, 6, \frac{1}{4}, 12)$	$(4, 2, 1, 12, \frac{1}{2}, 12)$

Table 4: Table of Clebsch coefficients for the lepton mass and mixing angle predictions defined in Eqs.59,62-65.

leading to the predictions

$$\begin{aligned}
|V_{\mu\tau}(M_X)| &= 2|V_{cb}(M_X)|, \quad |V_{\tau e}(M_X)| = 0, \quad |V_{e\tau}(M_X)| = 0, \\
|V_{e\mu}(M_X)| &= -\frac{x_{21}^E}{x_{21}^O} \frac{\lambda_c(M_X)}{\lambda_\mu(M_X)} \frac{|V_{ub}(M_X)|}{|V_{ub}(M_X)|}, \quad \lambda_{\nu_\mu}(M_X) = \frac{|V_{ub}(M_X)|}{|V_{cb}(M_X)|} \lambda_c(M_X) \quad . \quad (67)
\end{aligned}$$

Table 4 displays all of the Clebsch coefficients associated with the predictions in Eqs.59,62-65. Having constrained the leptonic mixing matrix elements at  $M_X$  for any particular ansatz  $A_i, B_j$  with the predictions in Eqs.65-62, the RGEs in Eq.39 are employed to yield low-energy predictions of the quantities. The mixing angle predictions are displayed in Table 5. Note that the mixing angles displayed in Table 5 are not significantly dependent upon  $M$  when compared to the uncertainties in the predictions. Part of the uncertainties of the predictions within one particular ansatz  $A_i, B_j$  are correlated by  $\alpha_S(M_Z), m_b$ . For reasons of brevity, we refrain from displaying these correlations for all possible ansatze. The table illustrates the powerful predictability (and therefore testability) of our scheme.

## 7 Filtering the Results

The muon and tau neutrino masses are both quite dependent upon (and are fixed by a choice of)  $M$ . However, the tau neutrino mass is relevant for cosmology, both to the cosmological bound[38]

$$m_{\nu_\tau} \leq \sum_{i=e,\mu,\tau} m_{\nu_i} < 100 \text{ eV}, \quad (68)$$

and from the point of view of structure formation in the early universe (recent monte-carlo simulations of which predict  $m_{\nu_\tau} \sim O(5) \text{ eV}$ ). We shall insist that the bound in

$B_{1,2,3}$	$A_{1,3,4}$	$A_{2,5}$	$A_7^K$	$A_7^O$	$A_8^K$	$A_8^O$
$ V_{\mu\tau} /10^{-1}$	0.32-0.55	0.95-1.64	0.63-1.09	0.63-1.09	0.24-0.41	0.63-1.09
$ V_{e\mu} /10^{-2}$	1.6-4.6	1.6-4.6	6.0-12.1	2.3-5.8	3.0-7.1	1.2-3.9
$ V_{e\tau} /10^{-3}$	0.5-1.4	1.6-4.3	4.2-11.4	1.6-4.3	0.8-2.1	0.8-2.1
$ V_{\tau e} /10^{-4}$	2.4-10.9	7.2-32.6	4.8-21.7	4.8-21.7	1.8-8.2	4.8-21.7

$B_{4,5,6}$	$A_{1,3,4}$	$A_{2,5}$	$A_7^K$	$A_7^O$	$A_8^K$	$A_8^O$
$ V_{\mu\tau} /10^{-1}$	0.32-0.55	0.95-1.64	0.63-1.09	0.63-1.09	0.24-0.41	0.63-1.09
$ V_{e\mu} /10^{-2}$	2.1-6.1	2.1-6.1	8.0-16.1	3.1-7.8	4.1-9.4	1.7-5.3
$ V_{e\tau} /10^{-3}$	0.7-1.9	2.1-5.7	5.6-15.2	2.1-5.7	1.1-2.8	1.1-2.8
$ V_{\tau e} /10^{-4}$	3.2-14.5	9.6-43.5	6.4-29.0	6.4-29.0	2.4-10.9	6.4-29.0

$B_{7,8}$	$A_{1,3,4}$	$A_{2,5}$	$A_7^K$	$A_7^O$	$A_8^K$	$A_8^O$
$ V_{\mu\tau} /10^{-1}$	0.32-0.55	0.95-1.64	0.63-1.09	0.63-1.09	0.24-0.41	0.63-1.09
$ V_{e\mu} /10^{-2}$	4.2-12.2	4.2-12.2	16.1-32.2	6.2-15.5	8.1-18.8	3.3-10.5
$ V_{e\tau} /10^{-3}$	1.4-3.8	4.2-11.4	11.2-30.3	4.2-11.4	2.1-5.7	2.1-5.7
$ V_{\tau e} /10^{-4}$	6.4-29.0	19.3-86.9	12.8-58.0	12.8-58.0	4.8-21.7	12.8-58.0

$A_7^N$	$B_{1,2,3}$	$B_{4,5,6}$	$B_{7,8}$
$ V_{\mu\tau} /10^{-1}$	0.63-1.09	0.63-1.09	0.63-1.09
$ V_{e\mu} /10^{-2}$	5.2-20.1	6.9-26.8	13.9-53.6
$ V_{e\tau} /10^{-3}$	0	0	0
$ V_{\tau e} /10^{-4}$	0	0	0

Table 5: Leptonic mixing angles predicted at  $M_W$  by the ansatze  $A_j$  combined with any of the light ansatze  $B_i$ . The ranges shown take into account the effects of varying  $\alpha_S(M_Z) = 0.1 - 0.13$ ,  $m_b = 4.1 - 4.4$  GeV. Changing the Majorana mass scale  $M$  does not have a significant effect when compared to the large uncertainties inherent in the predictions.

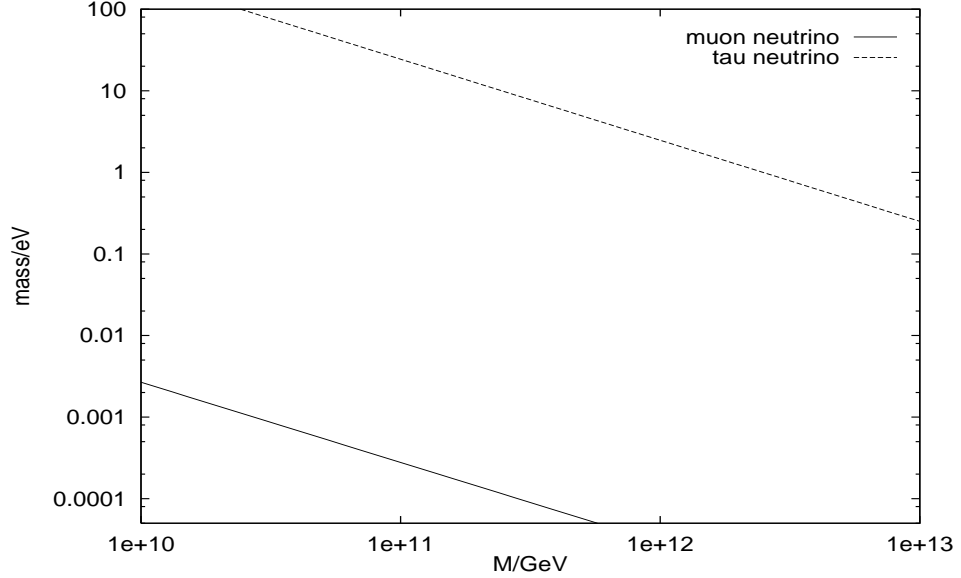


Figure 1: Dependence of  $m_{\nu_\tau}$  and  $m_{\nu_\mu}/l_N^2$  upon  $M$  for  $\alpha_S(M_Z) = 0.11$ ,  $m_b = 4.25$  GeV. Since the muon neutrino mass is scaled by  $1/l_N^2$ , the MSW bounds upon it implied by Eq.69 must also be scaled by this number. Thus, these bounds are dependent upon the choice of heavy ansatz  $A_i$  if applied to the figure.

Eq.68 is satisfied. This immediately places a lower bound[30] upon  $M$  of  $10^{10}$  GeV in quadruply unified third family Yukawa coupling scenarios.

As explained in the introduction, SUSY GUTs can provide an explanation for the solar neutrino deficit by providing values of  $\sin^2 2\theta_{e\mu}$  and  $\delta m_{e\mu}^2$  that are compatible with the MSW solution. To yield the prediction for  $m_{\nu_\mu}$ , the unification scale prediction in Eq.59 implied by picking some heavy ansatz  $A_i$  is run down to low energy using the RGE in Eq.39. The see-saw mechanism is then employed as in section 5 to obtain the mass of the light mass eigenstate of the muon neutrino at low energies. We shall constrain our predictions and possible ansatze by these constraints, which to the 95% confidence level are [4]

$$\delta m_{e\mu}^2 = 3 - 10 \times 10^{-6} \text{ eV}^2 \Rightarrow \delta m_{e\mu} = 1.7 - 3.2 \times 10^{-3} \text{ eV} \quad (69)$$

$$\sin^2 2\theta_{e\mu} = 8 - 150 \times 10^{-4} \Rightarrow |V_{e\mu}| = 1.4 - 6.1 \times 10^{-2}. \quad (70)$$

The mass constraint in Eq.69 is not compatible with ansatze  $(A_7^N, B_j)$  since these predict  $m_{\nu_\mu} = 0$  and so we discard these. For any of the other particular ansatze  $A_i, B_j$ , Eq.69 will be true only for some restricted range of  $M$ , as illustrated in Fig.1. This range of  $M$  is valid for some range of  $m_{\nu_\tau}$ , as displayed in Table 7. Note that in Table 7, the tau neutrino mass has been cut off at 100 eV to implement the cosmological bound. The results are valid for any  $B_i$  because the prediction of the muon neutrino mass in Eq.59 depends only upon the choice of heavy ansatz  $A_i$ , as is clear from the Clebsch coefficients in Table 4. Now we apply the MSW mixing angle

	$A_{1-5}$	$A_7^K$	$A_7^O$	$A_8^K$	$A_8^O$
$m_{\nu_\tau}/\text{eV}$	8-90	85-100	17-100	30-100	5-50
$M/10^{10} \text{ GeV}$	16-4	2-1	8-1	5-1	27-6

Table 6: Constraints on the tau neutrino mass and  $M$  implied by the MSW range of  $m_{e\mu}$  quoted in Eq.69 for each heavy ansatz  $A_i$ . The upper cut-off on  $m_{\nu_\tau} = 100$  eV implied by the cosmological bound in Eq.68 has been applied. The results are valid for  $\alpha_S(M_Z) = 0.1 - 0.13$ ,  $m_b = 4.1 - 4.4$  GeV.

filter in Eq.70 by checking that  $|V_{e\mu}|$  lies within the correct range. Table 5 shows that the ansatze  $(A_7^K, B_{4-8})$ ,  $(A_8^K, B_{7,8})$  predict a  $|V_{e\mu}|$  that lies outside the desired range and so these ansatze are discarded.

Next we constrain our results using the limits upon the  $\delta m_{\mu\tau}^2 - \sin^2 2\theta_{\mu\tau}$  parameter space given by the E531 experiment [13]. For  $\delta m_{\mu\tau}^2 > 100 \text{ eV}^2$ ,

$$\sin^2 2\theta_{\mu\tau} < 5 \times 10^{-3} \Rightarrow |V_{\mu\tau}| < 0.035. \quad (71)$$

Again, the choice of light ansatz  $B_j$  does not affect the prediction of  $|V_{\mu\tau}|$ . The predictions of  $|V_{\mu\tau}|$  displayed in Table 5 show that  $A_{2,5}, A_7^K, A_7^O, A_8^O$  are ruled out by this result. If one were to abandon the constraints implied by the MSW effect,  $M$  would be unconstrained and so if a larger value of  $M$  were chosen it would result in a smaller value of  $m_{\nu_\tau}$ . Thus  $M$  could always be chosen large enough to evade the E531 constraints. We require successful solutions to be compatible with the MSW range and so we do not consider these cases.

We are now left with successful ansatze  $(A_{1,3,4}, B_{1-8})$ ,  $(A_8^K, B_{1-6})$  which simultaneously satisfy the cosmological bounds, experimental constraints and which provide a solution to the solar neutrino problem.

## 8 Conclusions

We have generalised our previous operator analysis on charged fermion masses and quark mixing angles in the SUSY 422 model [23] to include neutrino masses and mixing angles. Since the model involves no large representations of the gauge group, and has no doublet-triplet splitting problem, the prospects for achieving string unification in this model are very good, and some attempts in this direction have already been made [22]. However here we have restricted ourselves to the low-energy effective field theory near the scale  $M_X \sim 10^{16}$  GeV, and parameterised the effects of string unification by non-renormalisable operators whose coefficients are suppressed by powers of  $(M_X/M_S)$ , where  $M_S > M_X$  is some higher scale associated with string physics.

It is worth emphasising that our previous operator analysis [23] uniquely specifies the Dirac sector of the neutrino matrices without any further assumptions or param-

eters. This is a simple consequence of quark-lepton unification and is also a feature of the SO(10) model. In order to make progress we have made the simple assumption of universal neutrino Majorana masses  $M$ . Since the Majorana masses are supposed to originate from some higher-dimensional operators in this model, as discussed in section 5, this implies some kind of flavour symmetry in the neutrino sector. The assumption of a right handed neutrino Majorana mass matrix proportional to the unit matrix in family space allows us to predict the magnitudes of all of the leptonic CKM matrix elements in terms of quark mixing angles and charged fermion masses, with the results being independent of  $M$  to good approximation. However the physical neutrino masses do depend on the parameter  $M$ , and we have chosen to fix  $M$  by requiring that the muon neutrino has a mass in the MSW range. Having done this the tau neutrino mass is then predicted, as shown in Table 6. In fact the magnitudes of all the leptonic CKM matrix elements are predicted (and are approximately independent of  $M$ ) as shown in Table 5.

We emphasise that the  $A_i, B_i$  in Eqs.18-25,27-34 were deduced from the known charged lepton masses and quark masses and mixing angles [23], and have fewer inputs than outputs in the charged fermion sector [23]. Third family Yukawa quadruple Yukawa unification (including the tau neutrino) leads to a prediction for  $m_t(\text{pole}) = 130 - 200$  GeV and  $\tan\beta = 35 - 65$ , depending on  $\alpha_S(M_Z)$  and  $m_b$ . Once the MSW bound on the muon neutrino mass was imposed,  $m_{\nu_\tau}$  is predicted to be in a range relevant for both the dark matter problem and structure formation in the early universe. More accurate predictions could be obtained if the error on  $\alpha_S(M_Z)$  and  $m_b$  were reduced. The high values of  $\tan\beta$  required by our model (also predicted in SO(10)) can be arranged by a suitable choice of soft SUSY breaking parameters as discussed in ref.[39], although this leads to a moderate fine tuning problem [40, 41]. This is discussed further in ref.[23].

As seen in Table 5 some of the operator combinations predict electron-muon neutrino mixing angles outside the MSW range, and some of them predict muon-tau mixing angles in conflict with current experimental limits. However some of the results are consistent with both, as summarised in Table 7. These results are associated with the Clebsch relations in Eqs. 26,35,59,62 which permits a qualitative understanding of the numerical results. For example for  $A_{1,3,4}$  ( $A_8^K$ ) we have  $\left(\frac{\lambda_\mu}{\lambda_s}\right)_{M_X} = 3(4)$ ,  $\left(\frac{\lambda_{\nu_\mu}}{\lambda_c}\right)_{M_X} = 3(3/2)$  and  $\left(\frac{|V_{\mu\tau}|}{|V_{cb}|}\right)_{M_X} = 1(3/4)$ . Thus once the muon neutrino mass has been adjusted into the correct MSW range, the Clebsch relations tell us that the tau neutrino mass predicted by  $A_8^K$  will be about four times larger than that predicted by  $A_{1,3,4}$ . Similarly, the level of muon-tau neutrino mixing is seen to be slightly larger in the case of  $A_{1,3,4}$  than for  $A_8^K$ , due to its slightly larger Clebsch. In fact the values of  $\delta m_{\tau\mu}^2$  and  $\sin^2 2\theta_{\mu\tau}$  quoted in Table 7 are on the edge of the present exclusion zone. If our scheme is correct, muon-tau neutrino oscillations will be seen very soon by the CHORUS and NOMAD experiments.

It is clear from Table 7 that  $A_8^K$  implies tau neutrino masses which are too high compared to the  $\sim 5$  eV masses suggested by recent calculations on structure formation in the early universe. However, ansatze  $A_{1,3,4}$  predict the tau neutrino mass to

	$A_{1,3,4}$	$A_8^K$
	$\delta m_{\mu\tau}^2 = 64 - 8100 \text{ eV}^2$ $\sin^2 2\theta_{\mu\tau} = 4.1 - 5.0 \times 10^{-3}$	$\delta m_{\mu\tau}^2 = 900 - 10^4 \text{ eV}^2$ $\sin^2 2\theta_{\mu\tau} = 2.3 - 5.0 \times 10^{-3}$
$B_{1,2,3}$	$\sin^2 2\theta_{e\mu} = 1.0 - 2.1 \times 10^{-3}$ $\sin^2 2\theta_{e\tau} = 1.1 - 8.1 \times 10^{-6}$ $\sin^2 2\theta_{\tau e} = 2.3 - 47.2 \times 10^{-7}$	$\sin^2 2\theta_{e\mu} = 3.6 - 20.0 \times 10^{-3}$ $\sin^2 2\theta_{e\tau} = 2.5 - 18.1 \times 10^{-6}$ $\sin^2 2\theta_{\tau e} = 1.3 - 26.6 \times 10^{-7}$
$B_{4,5,6}$	$\sin^2 2\theta_{e\mu} = 1.8 - 14.8 \times 10^{-3}$ $\sin^2 2\theta_{e\tau} = 2.0 - 14.4 \times 10^{-6}$ $\sin^2 2\theta_{\tau e} = 4.1 - 84.0 \times 10^{-7}$	$\sin^2 2\theta_{e\mu} = 6.7 - 35.9 \times 10^{-3}$ $\sin^2 2\theta_{e\tau} = 4.4 - 32.2 \times 10^{-6}$ $\sin^2 2\theta_{\tau e} = 2.3 - 47.3 \times 10^{-7}$
$B_{7,8}$	$\sin^2 2\theta_{e\mu} = 1.0 - 2.1 \times 10^{-3}$ $\sin^2 2\theta_{e\tau} = 7.8 - 57.4 \times 10^{-6}$ $\sin^2 2\theta_{\tau e} = 16.5 - 336 \times 10^{-7}$	Not in MSW range.

Table 7: A summary of our predictions which satisfy the MSW solar neutrino mass and mixing data, the cosmological tau neutrino mass upper bound and the E531 tau-muon neutrino mixing exclusion zone. The tau neutrino mass prediction depends on the parameter  $M$  which has been chosen so that the muon neutrino mass lies in the MSW range. The electron neutrino mass is effectively zero. The mixing angle predictions are approximately independent of  $M$ , although they do rely on the assumption of universality. Note that the  $\delta m_{\mu\tau}^2$  and  $\sin^2 2\theta_{\mu\tau}$  predictions are valid for each of the light ansatze  $B_i$ .

be as low as about 10 eV, a number in the right ball park for structure formation, so on this basis one may wish to discard the  $A_8^K$  ansatze. Restricting our attention to  $A_{1,3,4}$  only, we see from Table 3 that  $B_4$  predicts  $\left(\frac{\lambda_d}{\lambda_e}\right)_{M_X} = 3$  (the GJ prediction) while  $B_{1,5}$  predicts  $\left(\frac{\lambda_d}{\lambda_e}\right)_{M_X} = 4$ . The  $B_6, B_7$  are excluded by the down quark mass, and  $B_3, B_8$  give rather small down masses while  $B_2$  gives a rather large down mass and these may soon be excluded by more accurate measurements of the down quark mass and  $\alpha_3(M_Z)$  [23]. If one discounts these cases, then the list of possibilities is reduced to  $A_{1,3,4} \otimes B_{1,4,5}$ .

Table 7 shows that the combination of  $A_{1,3,4}$  with  $B_1$  involves  $\sin^2 2\theta_{e\mu} \approx 1 - 2 \times 10^{-3}$ , while  $A_{1,3,4}$  with  $B_{4,5}$  yields  $\sin^2 2\theta_{e\mu} \approx 2 - 15 \times 10^{-3}$ . These two possibilities may be therefore distinguished by data from future solar neutrino experiments, assuming the MSW mechanism to be operative. Since  $B_{4,5}$  give different down quark mass predictions, the combination of improved down quark mass determinations and solar neutrino data may eventually allow the particular ansatz  $B_i$  which is combined with  $A_{1,3,4}$  to be uniquely specified. The individual  $A_{1,3,4}$  cannot be differentiated by such a bottom-up procedure, however.

To summarise, in this paper we have exploited our recent operator analysis of the charged fermion sector [23] which fixes the Dirac part of the neutrino mass matrix, and supplemented it by the simple assumption of universal Majorana neutrino masses  $M$ . We have shown how the MSW solar neutrino data, plus the cosmological bound on the tau neutrino mass and the present limit on muon-tau neutrino mixing reduces a large number of possible ansatze to just a few. We find the idea of predicting all the neutrino masses and mixing angles in terms of the known fermion spectrum plus one additional parameter  $M$ , essentially a generalisation of the back-of-the-envelope estimate in Eq.2, to be a very attractive and simplifying idea. Within the framework of a particular model, we have shown how this idea of universal Majorana masses has strong predictive power in the neutrino sector, with imminently testable cosmological, astrophysical and terrestrial consequences.

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## Appendix

In our particular case  $A_1, B_1$  the Clebsch coefficients in Eq.50 are given by

$$\begin{aligned}
x_{12}^U &= 0 & x_{12}^D &= 1 & x_{12}^E &= 1 & x_{12}^N &= 0 \\
x_{21}^U &= 1 & x_{21}^D &= 3 & x_{21}^E &= 9/4 & x_{21}^N &= 3/4 \\
x_{22}^U &= 1 & x_{22}^D &= -1 & x_{22}^E &= 3 & x_{22}^N &= -3 \\
x_{22}'^U &= 1 & x_{22}'^D &= 1 & x_{22}'^E &= -3 & x_{22}'^N &= -3 \\
x_{32}^U &= 1 & x_{32}^D &= -1 & x_{32}^E &= -1 & x_{32}^N &= 1
\end{aligned} \tag{72}$$

and  $x_{12}'^U \neq 0$ .

At  $M_X$ , we have the freedom to rotate the phases of  $F^i$  and  $\bar{F}_j$ , since this leaves the lagrangian of the high energy theory invariant. In doing this we rotate away 5 phases in the matrices since there are only 5 *relative* phases:

$$\begin{aligned}
\begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} &\rightarrow \begin{bmatrix} e^{-i(\phi_{32}-\phi_{12})} & 0 & 0 \\ 0 & e^{-i(\phi_{32}-\phi_{22}^D)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} \\
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} &\rightarrow \begin{bmatrix} e^{-i(-\phi_{32}+\phi_{22}^D-\phi_{21})} & 0 & 0 \\ 0 & e^{i\phi_{32}} & 0 \\ 0 & 0 & e^{i\phi_{33}} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}.
\end{aligned} \tag{73}$$

Note that the first phase rotation performed in Eq.73 introduces phases into the Majorana mass matrix  $M \sim \text{diag}(e^{2i(\phi_{32}-\phi_{12})}, e^{2i(\phi_{32}-\phi_{22}^D)}, 1)$ .

Once the Yukawa matrices have been derived from the phase rotations in Eq.73,  $Y^{U,N}$  are made real by multiplying by the phase matrices

$$\begin{aligned}
Y^U &\rightarrow \begin{bmatrix} e^{-i\bar{\phi}_{12}} & 0 & 0 \\ 0 & e^{-i\bar{\phi}_{22}} & 0 \\ 0 & 0 & 1 \end{bmatrix} Y^U \begin{bmatrix} e^{i\bar{\phi}_{22}^U} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
Y^N &\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\bar{\phi}_{22}} & 0 \\ 0 & 0 & 1 \end{bmatrix} Y^N \begin{bmatrix} e^{i\bar{\phi}_{22}^U} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{74}$$

where we have defined  $\bar{\phi}_{22} \equiv \phi_{22}^U - \phi_{22}^D$  and  $\bar{\phi}_{12} \equiv \phi_{12}' - \phi_{12}$ . This amounts to a phase redefinition of the  $(U_R)^c$ ,  $U_L$ ,  $\nu_R$  and  $\nu_L$  fields.

To diagonalise the real matrices obtained from the above phase rotations, we first diagonalise the heavy 2 by 2 submatrices, then the light submatrices as shown below,

$$\begin{aligned}
Y^D &\rightarrow \begin{bmatrix} \tilde{c}_2 & \tilde{s}_2 & 0 \\ -\tilde{s}_2 & \tilde{c}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_4 & \tilde{s}_4 \\ 0 & -\tilde{s}_4 & \tilde{c}_4 \end{bmatrix} Y^D \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{c}_4 & -\bar{s}_4 \\ 0 & \bar{s}_4 & \bar{c}_4 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
Y^U &\rightarrow \begin{bmatrix} \tilde{c}_1 & \tilde{s}_1 & 0 \\ -\tilde{s}_1 & \tilde{c}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_3 & \tilde{s}_3 \\ 0 & -\tilde{s}_3 & \tilde{c}_3 \end{bmatrix} Y^U \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{c}_3 & -\bar{s}_3 \\ 0 & \bar{s}_3 & \bar{c}_3 \end{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$



$$\begin{aligned}
Y^E &\rightarrow \begin{bmatrix} \tilde{c}_2^l & \tilde{s}_2^l & 0 \\ -\tilde{s}_2^l & \tilde{c}_2^l & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_4^l & \tilde{s}_4^l \\ 0 & -\tilde{s}_4^l & \tilde{c}_4^l \end{bmatrix} & Y^E &\begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_4^l & -\tilde{s}_4^l \\ 0 & \tilde{s}_4^l & \tilde{c}_4^l \end{bmatrix} \begin{bmatrix} c_2^l & -s_2^l & 0 \\ s_2^l & c_2^l & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
Y^N &\rightarrow \begin{bmatrix} \tilde{c}_1^l & \tilde{s}_1^l & 0 \\ -\tilde{s}_1^l & \tilde{c}_1^l & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_3^l & \tilde{s}_3^l \\ 0 & -\tilde{s}_3^l & \tilde{c}_3^l \end{bmatrix} & Y^N &\begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_3^l & -\tilde{s}_3^l \\ 0 & \tilde{s}_3^l & \tilde{c}_3^l \end{bmatrix} \begin{bmatrix} c_1^l & -s_1^l & 0 \\ s_1^l & c_1^l & 0 \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{75}$$

where  $c_i^{(l)}, \bar{s}_i^{(l)}$  refer to  $\cos \theta_i^{(l)}$  and  $\sin \bar{\theta}_i^{(l)}$  respectively. Note that since  $Y^{U,D,E,N}$  are not symmetric  $\tilde{c}_i^{(l)}, \tilde{s}_i^{(l)}$  are independent of  $c_i^{(l)}, \bar{s}_i^{(l)}$ .

The quark mixing angles are given by  $\bar{s}_4 = -x_{32}^D H_{32}/H_{33}$ ,  $s_2 = -x_{21}^D H_{21}/(\lambda_s)_{M_X}$ ,  $s_1 = -x_{21}^U H_{21}/(\lambda_c)_{M_X}$  and  $\bar{s}_3 = -x_{32}^U H_{32}/H_{33}$ . Note that in the limit  $O_{12}' \rightarrow 0$  the up quark is massless in the model because it is generated by the small operator  $(\lambda_u)_{M_X} = -x_{12}'^U x_{21}^U H_{12}' H_{21}/(\lambda_c)_{M_X}$ . The lepton mixing angles are  $\bar{s}_4^l = -x_{32}^E H_{32}/H_{33}$ ,  $s_2^l = -x_{21}^E H_{21}/(\lambda_\mu)_{M_X}$ ,  $s_1^l = -x_{21}^N H_{21}/(\lambda_{\nu_\mu})_{M_X}$  and  $\bar{s}_3^l = -x_{32}^N H_{32}/H_{33}$ . We also denote  $\theta_3^{(l)} = \bar{\theta}_3^{(l)} - \bar{\theta}_4^{(l)}$  and  $\phi = -\bar{\phi}_{22}$ .

Noting that  $|V_{\mu\tau}| \sim s_3^l = (x_{32}^N - x_{32}^E)H_{32}/H_{33}$  and that  $|V_{cb}| \sim s_3 = (x_{32}^U - x_{32}^D)H_{32}/H_{33}$  we make the prediction in Eq.62 where  $m = (x_{32}^N - x_{32}^E)/(x_{32}^U - x_{32}^D)$  in this case. We also have that  $|V_{e\tau}| \sim s_1^l s_3^l \Rightarrow |V_{e\tau}|/|V_{\mu\tau}| = s_1^l = -x_{21}^N H_{21}/\lambda_{\nu_\mu}$ . From  $V_{CKM}$ ,  $H_{21} = -|\lambda_c||V_{ub}|/(x_{21}^u|V_{cb}|)$ . Substituting Eqs.62,59 yields Eq.63 where  $n = mx_{21}^N/(x_{21}^U l_N)$ . When  $H_{21}$  is substituted into the ratio  $|V_{\tau e}/V_{\mu\tau}| \sim s_2^l = x_{21}^E H_{21}/\lambda_\mu$ , we obtain our next prediction as in Eq.64. Our final prediction is for  $|V_{e\mu}| \sim |-s_2^l e^{i\phi} + s_1^l|$  which is made in terms of the phase  $\phi$ . As above, we substitute the angles  $\theta_1^l, \theta_2^l$  to obtain Eq.65 where  $p = x_{21}^N/(l_N x_{21}^U)$ ,  $q = l_N x_{21}^E/x_{21}^N$ .

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